MATHEMATICAL EMANCIPATIONS.

THE PASSING OF THE POINT AND THE NUMBER THREE: DIMENSIONALITY AND HYPERSPACE.

AMONG the splendid generalizations effected by modern mathematics, there is none more brilliant or more inspiring or more fruitful, and none more nearly commensurate with the limitless immensity of being itself, than that which produced the great concept variously designated by such equivalent terms as hyperspace, multidimensional space, n-space, n-fold or n-dimensional space, and space of n dimensions.

In science as in life the greatest truths are the simplest. Intelligibility is alike the first and the last demand of the understanding. Naturally, therefore, those scientific generalizations that have been accounted really great, such as the Newtonian law of gravitation, or the principle of the conservation of energy, or the all-conquering concept of cosmic evolution, are, all of them, distinguished by their simplicity and apprehensibility. To that rule the notion of hyperspace presents no exception. For its fair understanding, for a live sensibility to its manifold significance and quickening power, a long and severe mathematical apprenticeship, however helpful it would be, is not demanded in preparation, but only the serious attention of a mature intelligence reasonably inured by discipline to the exactions of abstract thought and the austerities of

the higher imagination. And it is to the reader having this general equipment, rather than to the professional mathematician as such, that the present communication is addressed.

To a clear understanding of what the mathematician means by hyperspace, it is in the first place necessary to conceive in its full generality the closely related notion of dimensionality and to be able to state precisely what is meant by saying that a given manifold has such and such a dimensionality, or such and such a number of dimensions, in a specified entity or element.

Discrimination, as the proverb rightly teaches, is the beginning of mind. The first psychic product of that initial psychic act is *numerical*: to discriminate is to produce *two,* the simplest possible example of multiplicity. The discovery, or better the invention, better still the production, best of all the creation, of multiplicity with its correlate of number, is, therefore, the most primitive achievement or manifestation of mind. Such creation is the immediate issue of intellection, nay, it *is* intellection, identical with its deed, and, without the possibility of the latter, the former itself were quite impossible. Accordingly it is not matter for surprise but is on the contrary a perfectly natural or even inevitable phenomenon that explanations of ultimate ideas and ultimate explanations in general should more and more avail themselves of analytic as distinguished from intuitional means and should tend more and more to assume arithmetic form. Depend upon it, the universe will never be really understood unless it may be sometime resolved into an ordered multiplicity and made to own itself an everlasting drama of the calculus.*

Let us, then, trust the arithmetic instinct as funda-

* In this connection I take pleasure in referring the reader to the somewhat naïve indeed but none the less very suggestive and stimulating article "The Warp of the World" by Mr. Newman Howard, *Hibbert Journal,* January, 1905.

mental and, for instruments of thought that shall not fail, repair at once to the domain of number. Every one who may reasonably aspire to a competent knowledge of the subject in hand is more or less familiar with the system of real numbers, composed of the positive and negative integers and fractions, such irrational numbers as $\sqrt{2}$ and π and countless hosts of similar numbers similarly definable. He may know that, for reasons which need not be given here, the system of real numbers is commonly described as the analytical continuum of first order. He knows, too, at any rate it is a fact which he will assume and readily appreciate, that the distance between any two points of a right line is exactly expressible by a number of the continuum; that, conversely, given any number, two points may be found whose distance apart is expressed by the numerical value of that number; that, therefore, it is possible to establish a unique and reciprocal, or one-to-one, correspondence between the real numbers and the points of a straight line, namely, by assuming some point of the line as a fixed point of reference or origin of distances, by agreeing that a distance shall be positive or negative according as it proceeds from the origin in this sense or in the other and by agreeing that a *point* and the *number* which by its magnitude reckoned in terms of a chosen finite unit however great or small serves to express the distance of the point from the origin and by its sign indicates on which side of the origin the point is situated, shall be a pair of *correspondents*. Accordingly, if the point P glides along the line, the corresponding number v will vary in such a way that to each position of the geometric there corresponds one value of the arithmetic element, and conversely. P represents v; and v, P. No two P's represent a same v; and no two v's, a same P. By virtue of the correlation thus established with the *analytical* continuum, we may describe the line as a simple

or one-fold *geometric* continuum, namely, of points. The like may in general be said, and for the same reason, of any curve whatever, but we select the straight line as being the simplest, for in matters fundamental we should prefer clearness to riches of illustration, in the faith that, if first we seek the former, the latter shall in due course be added unto it. The straight line, when it is regarded as the domain of geometric operation, as the region or room containing the configurations or sets of elements with which we deal, is and is called a *space;* and this space, viewed as the manifold or assemblage of its points, is said to be *one*-dimensional for the reason that, as we have seen, in order to determine the position of a point in it, in order, i. e., to pick out or distinguish a point from all the other points of the manifold, it is necessary and sufficient to know one fact about the point, as, e. g., its distance from an assumed point of reference. In other words, the line is called a one-dimensional space of points because in that space the point has one and but one degree of freedom or, what is tantamount, because the position of the point depends upon the value of a single v, known as its coordinate.

Herewith is immediately suggested the generic concept of dimensionality: *if an assemblage of elements of any given kind whatsoever, geometric or analytic or neither, as points, lines, circles, triangles, numbers, notions, sentiments, hues, tones, be such that, in order to distinguish every element of the assemblage from all the others, it is necessary and sufficient to know exactly n independent facts about the element, then the assemblage is said to be n-dimensional in the elements of the given kind.* It appears, therefore,that the notion of dimensionality is by no means exclusively associated with that of space but on the contrary may often be attached to the far more generic concept of assemblage, aggregate or manifold. For ex-

ample, duration, the total aggregate of time-points, or instants, is a simple or one-fold assemblage. On the other hand, the assemblage of colors is three-dimensional as is also that of musical notes, for in the former case, as shown by Clerk Maxwell, Thomas Young and others, every color is composable as a definite mixture of three primary ones and so depends upon three independent variables or coordinates expressing the amounts of the fundamental components. And in the latter case a similar scheme obtains, one note being distinguishable from all others when and only when the three general marks, pitch, length, and loudness, are each of them specified. In passing it seems worth while to point out the possibility of appropriating the name *soul* to signify the *manifold* of all *possible* psychic *experiences,* in which event the term would signify an assemblage of probably *infinite* dimensionality, and the assemblage would be continuous, too, if Oswald* be right in his contention that every manifold of experience possesses the character of continuity. That contention, however, if the much abused term continuity be allowed to have its single precise definitely seizable scientific meaning, is far less easy to make good than that eminent chemist and courageous philosopher seems to think.

Returning to the concept of space, an n-fold *assemblage* will be an n-dimensional *space* if the elements of the assemblage are geometric entities of any given kind. We have seen that the straight line is a *one*-dimensional space of *points*. But in studying the right line conceived as a space, we are not compelled to employ the point as element. Instead we may choose to assume as element the point *pair* or *triplet* or *quatrain,* and so on. The line would then be for our thought primarily a space, not of points, but of point pairs or triplets and so on, and it would accordingly be strictly a space of *two dimensions* or of *three,*

* Cf. his *Natur-Philosophie.*

and so on; for, obviously, to distinguish say a point pair from all other such pairs we should have to know *two* independent facts about the pair, one of which would determine one of the points, and the other, the other. The pair would have two degrees of freedom in the line, its determination would depend upon two independent variables as v_1 and v_2 referred to two (in general different) points of reference. It should be signalized that a pair of points is not merely two points, nor a triplet merely three. *Order* is essential, and two points will furnish two distinct pairs, and three points will furnish six pairs or six triplets; on the other hand, three pairs or three triplets may contain but three points. On its arithmetic side the shield presents a precisely parallel doctrine. The simple analytical continuum composed of the real numbers immediately loses its simplicity and assumes the character of a 2- or 3- . . . or n-fold analytical continuum if, instead of thinking of its individual numbers, we view it as an aggregate of number pairs or triplets or, in general, as the totality of ordered systems of n numbers each.

In the light of the preceding paragraph it is seen that the dimensionality of a given space is not unique but depends upon the choice of geometric entity for primary or generating element. A space being given, its dimensionality is not therewith determined but depends upon the will of the investigator, who by a proper choice of generating element may endow the space with any dimensionality he pleases. That fact is of cardinal significance alike for science and for philosophy. I reserve for a little while its further consideration in order to present at once a kind of complementary fact of equal interest and of scarcely less importance. It is that two spaces which in every other respect are essentially unlike, thoroughly disparate, may, by suitable choice of generating elements, be made to assume equal dimensionalities. Consider, for example,

the totality of lines contained in a same plane and containing a point in common. Such a totality, called a *pencil*, of lines is a simple geometric continuum, namely, of lines. It is, then, and may be called, a *one*-dimensional space of *lines* just as the line or *range* of points is a one-dimensional space of points. The two spaces are equally rich in their respective elements. And if, following Desargues and his successors, we adjoin to the points of the range a so-called "ideal" point or point at infinity, thus rendering the range like the pencil, *closed*, it becomes obvious that two intelligences, adapted and confined respectively to the two simple spaces in question, would enjoy equal freedom; their analytical experiences would be identical, and their geometries, though absolutely disparate in kind, would be equally rich in content. Just as the range-dweller would discover that the dimensionality of his space is *two* in *point pairs, three* in *triplets,* and so on, so the pencil-inhabitant would find his space to be of dimensionality *two* in *line pairs, three* in *triplets,* and so on without end. It was indicated above that *any* line, straight or *curved,* is a *one*-dimensional space of *points.* In that connection it remains to say that, speaking generally, *any curve,* literally and strictly conceived as the assemblage of its (tangent) lines and so including the point or pencil as a special case, is also a *one*-dimensionel space of *lines.* It is, moreover, obvious that the foregoing considerations respecting the range of points and the pencil of lines are, *mutatis mutandis,* equally valid for any one of an infinite variety of other analogous spaces, as, e. g., the *axal pencil,* a one-fold space of planes, consisting of the totality of planes having a line in common.

If perchance some reader should feel an ungrateful sense of impropriety in our use of the term *space* to signify such common geometric aggregates as we have been considering, I gladly own that his state of mind is a per-

fectly natural one. But it is, besides and on that account, a source of real encouragement. Dictional sensibility is a hopeful sign, being conclusive evidence of life, and, while there is life, there remains the possibility and therewith the hope of readjustment. In the present case, I venture to assure the reader, on grounds both of personal experience and of the experience of others, that whatever sense he may have of injury received will speedily disappear in the further course of his meditations. Only, let him not be impatient. Larger meanings must have time to grow; the smaller ones, those that are most natural and most provincial, being also the most persistent. In the process of clarification, expansion and readjustment, his fine old word, space, early come into his life and gradually stained through and through with the refracted partial lights and multi-colored prejudices of his youth, is not to be robbed of its proper charms nor to be shorn of its proper significance. More than it will lose of mystery, it shall gain of meaning. Of this last it has hitherto had for him but little that was of scientific value, but little that was not vague and elusive and ultimately unseizable. That was because the word stood for something absolutely *sui generis, i. e., for a genus neither including species nor being itself included in a class.* But now, on the other hand, both of these negatives are henceforth to be denied, and the hitherto baffling term, perfect symbol of the unthinkable, always promising and never presenting definable content, immediately assumes the characteristic twofold aspect of a genuine concept, being at once included as member of a higher class, the more generic class of manifolds, and including within itself an endless variety of individuals, an infinitude of species of space.

Of these species, the next, in order of simplicity, to those above considered, is the plane. To distinguish a point of a plane from all its other points, it is necessary

and sufficient to know *two* independent facts about its position, as, e. g., its distances from two assumed lines of reference, most conveniently taken at right angles. Viewed as the *ensemble* of its points, the plane is, therefore, a space of *two* dimensions. In that space, the point enjoys a freedom exactly twice that of a point in a range or of a line in a pencil, and exactly equal to that of a *pair* of points or of lines in the last-mentioned spaces. On the other hand, if the point *pair* be taken as element of the plane, the latter becomes a space of *four* dimensions.

What if the *line* be taken as generating element? It is obvious that the plane is equally rich in pencils and in ranges. It contains as many lines as points, neither more nor less. Two points determine a line; two lines, a point; if the lines be parallel, their common point is a Desarguesian, a point at infinity. We should therefore expect to find that in a plane the position of a line depends upon two and but two independent variables. And the expectation is realized, as it is easy to see. For if the variables be taken to represent (say) distances measured from chosen points along two lines of reference, it is immediately evident that a given pair of values of the variables determines a line uniquely and that, conversely, a given line uniquely determines such a pair. The plane is, therefore, a *two*-dimensional space of *lines* as well as of points. In line *pairs*, as in point pairs, its dimensionality is *four*. We may suppose the space in question to be inhabited by two sorts of individuals, one of them capable of thinking in terms of points but not of lines, the other in terms of lines but not of points. Each would find his space bi-dimensional. They would enjoy precisely the same analytical experience. Between their geometries there would subsist a fact-to-fact *correspondence* but not the slightest *resemblance*. For example, the circle would be for the former individual a certain assemblage of points but devoid of

tangent lines, and, for the latter, a corresponding assemblage of (tangent) lines but devoid of contact points.

Passing from the plane to a curved surface, to a sphere, for example, a little reflection suffices to show that the latter may be conceived in a thousand and one ways, most simply as the ensemble of its points *or* of its (tangent) planes *or* of its (tangent) lines. These various concepts are logically equivalent and in themselves are equally intelligible. And if to us they do not seem to be also equally *good,* that is doubtless because we are but little traveled in the great domain of Reason and therefore naturally prefer our familiar customs and provincial points of view to others that are strange. At all events, it is certain that on purely *rational* grounds, none of the concepts in question is to be preferred, while, from preference based on *other* grounds, it is the office alike of science and of philosophy to provide the means of emancipation. Let us, then, detach ourselves from the vulgar point of view and for a moment contemplate the three concepts as coordinate indeed but independent concepts of surface. And for the sake of simplicity, we may think of a sphere.* Suppose it placed upon a plane and imagine its highest point, which we may call the pole, joined by straight lines to all the points of the plane. Each line pierces the sphere in a second point. It is plain that thus a one-to-one correspondence is set up between the points of the sphere and those of the plane, except that the pole corresponds at once to all the Desarguesian points of the plane—an exception, however, which is here of no importance. The plane and the sphere are, then, equally rich in points. Accordingly, the sphere conceived as a plenum or locus of *points* is a space of *two* dimensions. In that space the point has two degrees of freedom. Its position depends upon two independent vari-

* The term is here employed as in the higher geometry to denote, not a solid, but a surface.

ables, as latitude and longitude. But we may conceive the surface quite otherwise: at each of its points there is a (tangent) plane, and now, *disregarding points,* we may think only of the assemblage of those planes. These together constitue a sphere, not, however, as a locus of points, but as an envelope (as it is called) of planes. And what shall we say of the surface as thus conceived? The answer obviously is that it is a *two*-dimensional space of *planes,* admitting of a geometry quite as rich and as definite as is the theory of any other space of equal dimensionality. In each of the planes there is a pencil of lines of which each is tangent to the sphere. Thus we are led to a third conception of our surface. We have merely to disregard both points and planes and confine our attention to the assemblage of lines. The vision which thus arises is that of a *three*-dimensional space of *lines.* In pencils, its dimensionality is two. In this space the pencil has two and the line three degrees of freedom.

But let us return to the plane. We have seen that at the geometrician's bidding it plays the rôle of a twofold space either in points or in lines. It is natural to ask whether it may be conceived as a space of *three* dimensions, like the sphere in its third conception. The answer is affirmative: it may be so conceived, and that in an infinity of ways. Of these the simplest is to assume the *circle* as primary or generating element. Of circles the plane contains a threefold infinity, an infinity of infinities of infinity. It is a circle continuum of third order. To distinguish any one of its circles from all the rest, three independent data, two for position and one for size, are necessary and sufficient. In the plane the circle has three degrees of freedom, its determination depends upon three independent variables. The plane is, accordingly, a tridimensional space of *circles.* In *parabolas* its dimensionality is *four;* in *conics, five;* and so on *without limit.*

Before turning to space, ordinarily so-called, it seems worth while to indicate another geometric continuum which, although it presents no *likeness* whatever to the plane, nevertheless *matches* it perfectly in every conceptual aspect. The reference is to the *sheaf,* or *bundle,* of *lines,* i. e., the totality of lines having a point in common. The point is to be disregarded and the lines viewed as non-decomposable entities, like points in a line or a plane regarded as an assemblage of points. Thus conceived, the *sheaf* is literally a *space,* namely, of *lines.* It is, in the vulgar sense of the term, just as *big,* occupies precisely as much room, nay indeed the same room, as the space in which we live. The sheaf as a space is *two*-dimensional in *lines,* like the *plane* in *points; two*-dimensional in *pencils,* like the *plane* in *lines; four*-dimensional in *line* or *pencil pairs,* like the *plane* in *point* or *line pairs; three*-dimensional in ordinary *cones,* like the *plane* in *circles;* and so on and on.

In the light of the foregoing considerations, any hitherto uninitiated reader will probably suspect that ordinary space is *not,* as it is commonly supposed and said to be, an *inherently* and *uniquely three*-dimensional affair. His suspicion is completely justified by fact. The simple traditional affirmation of tri-dimensionality is devoid of definite meaning. It is unconsciously elliptic, requiring for its completion and precision the specification of an appropriate geometric entity for generating element. Merely to say that space is tri-dimensional because a solid, e. g., a plank, has length, breadth and thickness, is too crude for scientific purposes. Moreover, it betrays, quite unwittingly indeed as we shall see, an exceedingly meager point of view. Not only does it assume the *point* as element but it does so tacitly because unconsciously, as if the point were not merely *an* but *the* element of ordinary space. *An* element the point may obviously be taken to be, and in

that element ordinary space is indeed tri-dimensional, for the position of a point at once determines and is determined by three independent data, as its distances from three assumed mutually perpendicular planes of reference. It must be admitted, too, that the point does, in a sense, recommend itself as the element *par excellence,* at least for practical purposes. For example, we prefer to do our drawing with the point of a pencil to doing it with a straight edge. But that is a matter of physical as distinguished from rational convenience. Preference for the point has, then, a cause: in the order of evolution, practical man precedes man rational and determines for the latter his initial choices. Causes, however, are extra-logical things, and the preference in question, though it has indeed a cause, has no reason. Accordingly, when in these modern times, the geometrician became clearly conscious that he was in fact and had been from time immemorial employing the point as element and that it was this use that lent to space its traditional triplicity of dimensions, he did not fail to perceive almost immediately the logically equal possibility of adopting at will for primary element any one of an infinite variety of other geometric entities and so the possibility of rationally endowing ordinary space with any prescribed dimensionality whatever.

Thus, for example, the plane is no less available for generating element than is the point. The plane is logically and intuitionally just as simple, for, if from force of habit, we are tempted to analyze the plane into an assemblage of points, the point is in its turn equally conceivable as or analyzable into an assemblage of planes, the sheaf of planes containing the point. We may, then, regard our space as primarily a plenum of planes. To determine a plane requires three and but three independent data, as, say, the distances to it measured along three chosen lines from chosen points upon them. It follows that ordinary

space is *three*-dimensional in *planes* as well as in points. But now if (with Plücker) we think of the *line* as element, we shall find that our space has *four* dimensions. That fact may be seen in various ways, most easily perhaps as follows. A line is determined by any two of its points. Every line pierces every plane. By joining the points of one plane to all the points of another, all the lines of space are obtained. To determine a line it is, then, enough to determine two of its points, one in the one plane and one in the other. For each of these determinations, two data, as before explained, are necessary and sufficient. The position of the line is thus seen to depend upon four independent variables, and the four-dimensionality of our space in lines is obvious. Again, we may (with Lie) view our space as an assemblage of its *spheres*. To distinguish a sphere from all other spheres, we need to know four and but four independent facts about it, as, say, three that shall determine its center and one its size. Hence our space is *four*-dimensional also in *spheres*. In *circles* its dimensionality is *six;* in *surfaces* of *second order* (those that are pierced by a straight line in two points), *nine*; and so on *ad infinitum.*

Doubtless the reader is prepared to say that, if the foregoing account of hyperspace be correct, the notion is after all a very simple one. Let him be assured, the account is correct and his judgment is just: the notion is simple. That property, as said in the beginning, is indeed one of its merits. As presented the concept is entirely free from mystery. To seize upon it, it is unnecessary to pass the bounds of the visible universe or to transcend the limits of intuition. Its realization is found even in the line, in the pencil, in the plane, in the sheaf, here, there and yonder, everywhere, in fact. The account, however, though quite correct, is not yet complete. The term hyperspace has yet another meaning and yet in strictness not

another, as we shall see. It will be noticed that among the foregoing examples of hyperspace, none is presented of dimensionality *exceeding three* in *points.* It is precisely this variety of hyperspace that the term is commonly employed to signify, particularly in popular enquiry and philosophical speculation. And it is this variety, too, that just because it baffles the ordinary visual imagination, proves to be, for the non-mathematician at any rate, at once so tantalizing, so mysterious and so fascinating.

It remains, then, to ask, what is meant by a hyperspace of *points?* How is the notion formed and what is its motivity and use? The path of enquiry is a familiar one and is free from logical difficulty. Granted that a one-to-one correspondence can be established between the real numbers and the points of a right line, so that the geometric serve to represent the arithmetic elements; granted that all (ordered) pairs of numbers are similarly representable by the points of a plane, and all (ordered) triplets by the points of ordinary space; the suggestion then naturally presents itself that, whether there really is or not, there *ought* to be a *space* whose *points* would serve to *represent,* as in the preceding cases, *all ordered systems* of *values* of *n independent variables;* and especially to an analyst with a strong geometric predilection, to one who is a born *Vorstellender,* for whom analytic abstractions naturally tend to take on figure and assume the exterior forms of sense, that suggestion comes with a force which he alone perhaps can fully appreciate. And what does he do? Not finding the desiderated hyperspace present to his *vision* or *intuition* or *visual imagination,* he *posits* it, or if you prefer, he *creates* it, *in thought.* The concept of hyperspace of points is thus seen to be off-spring of Arithmetic and Geometry. It is legitimate fruit of the indissoluble union of the fundamental sciences.

Does such hyperspace exist? It does exist genuinely.

If not for intuition, it exists for conception; if not for imagination, it exists for thought; if not for sense, it exists for reason; if not for matter, it exists for mind. These if's are *if*'s in fact. The question of imaginability is really a question. We shall return to it presently.

The concept of hyperspace of points is generable in various other ways. Of all ways the following is perhaps the best because of its appeal at every stage to intuition. Let there be two points and grant that these determine a *line,* point-space of *one* dimension. Next posit a point *outside* of this line and suppose it joined by lines to all the points of the given line. The points of the joining lines together constitute a *plane,* point-space of *two* dimensions. Next posit a point outside of this plane and suppose it joined by lines to all the points of the plane. The points of all the joining lines together constitute an ordinary *space,* point-space of *three* dimensions. The clue being now familiar to our hand, let us boldly pursue the opened course. Let us overleap the limits of common imagination, transcend ordinary intuition as being at best but a non-essential auxiliary, and in *thought* posit an extra point that, *for thought* at all events, shall be *outside* the space last generated. Suppose that point joined by lines to all the points of the given space. The points of the joining lines together constitute a point-space of *four* dimensions. The process here applied is perfectly clear and obviously admits of endless repetition.

Moreover, the process is equally available for generating hyperspaces of other elements than points. For example, let there be two intersecting lines and grant that these determine a *pencil,* line-space of *one* dimension. Next posit a line (through the vertex) *outside* of the given pencil and suppose it joined by pencils to all the lines of the given pencil. The lines of the joining pencils together constitute a *sheaf,* line-space of *two* dimensions. Next

posit a line (through the vertex) *outside* of the sheaf and suppose it joined by pencils to all the lines of the sheaf. The lines of the joining pencils constitute a *hypersheaf,* line-space of *three* dimensions. The next step plainly leads to a line-space of *four* dimensions; and so on *ad infinitum.*

And now as to the question of imaginability. Is it possible to intuit configurations in a hyperspace of points? Let it be understood at the outset that that is not in any sense a mathematical question, and mathematics as such is quite indifferent to whatever answer it may finally receive. Neither is the question primarily a question of philosophy. It is first of all a psychological question. Mathematicians, however, and philosophers are also men and they may claim an equal interest perhaps with others in the profounder questions concerning the potentialities of our common humanity. The question, as stated, undoubtedly admits of affirmative answer. For the lower spaces, with which the imagination *is* familiar, exist *in* the higher, as the line in the plane, and the plane in ordinary space. But that is not what the question means. It means to ask whether it is possible to imagine hyper-configurations of points, i. e., point-configurations that are not wholly contained in a point-space (like our own) of three dimensions. It is impossible to answer with absolute confidence. One reason is that the term imagination still awaits precision of definition. Undoubtedly just as three-dimensional figures may be represented in a plane, so four-dimensional figures may be represented in space. That, however, is hardly what is meant by imagining them. On the other hand, a four-dimensional figure may be rotated and translated in such a way that all of its parts come one after another into the threefold domain of the ordinary intuition. Again, *the structure of a fourfold figure, every minutest detail of its anatomy, can be traced out by analogy with its three-dimensional analogue.* Now in such

processes, repetition yields skill, and so they come ultimately to require only amounts of *energy* and of *time* that are quite *inappreciable*. Such skill once attained, the *parts* of a familiar *fourfold configuration* may be made to pass before the eye of intuition in such *swift* and *effortless* succession that the configuration *seems present as a whole in a single instant*. If the *process* and *result* are not, properly speaking, *fourfold imagination* and *fourfold image*, it remains for the psychologist to indicate what is lacking.

Certainly there is naught of absurdity in supposing that under suitable stimulation the human mind may in course of time even speedily develop a spatial intuition of four or more dimensions. At present, as the psychologists inform us and as every teacher of geometry discovers independently, the spacial imagination, in case of very many persons, comes distinctly short of being strictly even tridimensional. On the contrary, it is flat. It is not every one, even among scholars, that with eyes closed can readily form a visual image of the *whole* of a simple *solid* like a sphere, enveloping it completely with bent beholding rays of psychic light. In such defect of imagination, however, there is nothing to astonish. In the first place, man as a race is only a child. He has been on the globe but a little while, long indeed compared with the fleeting evanescents that constitute the most of common life, but very short, the merest fraction of a second, in the infinite stretch of time. In the second place, circumstances have not, in general, favored the development of his higher potentialities. His chief occupation has been the destruction and evasion of his enemies, contention for mere existence against hostile environment. Painful necessity, then, has been the mother of his inventions. That, and not the vitalizing joy of self-realization, has for the most part determined the selection of the fashion of his faculties.

But it would be foolish to believe that these have assumed their final form or attained the limits of their potential development. The imperious rule of necessity will relax. It will never pass quite away but it will relax. It is relaxing. It has relaxed appreciably. The intellect of man will be correspondingly quickened. More and more will joy in its activity determine its modes and forms. The hyper-dimensional worlds that man's reason has already created, his imagination may yet be able to depict and illuminate.

It remains to ask, finally, what purpose the concept of hyperspace subserves. Reply, partly explicit but chiefly implicit, is not, I trust, entirely wanting in what has been already said. Motivity, at all events, and *raison d'être* are not far to seek. On the one hand, the great generalization has made it possible to enrich, quicken and beautify analysis with the terse, sensuous, artistic, stimulating language of geometry. On the other hand, the hyperspaces are in themselves immeasurably interesting and inexhaustibly rich fields of research. Not only does the geometrician find light in them for the illumination of many otherwise dark and undiscovered properties of the ordinary spaces of intuition, but he also discovers there wondrous structures quite unknown to ordinary space. These he examines. He handles them with the delicate instruments of his analysis. He beholds them with the eye of the understanding and delights in the presence of their supersensuous beauty.

It is by creation of hyperspaces that the rational spirit secures release from limitation. In them it lives ever joyously, sustained by an unfailing sense of infinite freedom.

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